TRIVALENT DIAGRAMS, MODULAR GROUP AND TRIANGULAR MAPS

SAMUEL VIDAL [1]

[1] Laboratoire d'Informatique fondamentale de Lille (LIFL)
Université de Science et Technologies de Lille, Bâtiment M2
59655 Villeneuve d'Ascq, France.
E-mail address: samuel.vidal@math.univ-lille1.fr
URL: http://www.lifl.fr
URL: http://www.samuel.math.online.fr

Abstract. The aim of the paper is to give a preliminary overview of some of the results of the thesis prepared by the author. We propose a bijective classification of the subgroups of the modular group by pointed trivalent diagrams. Conjugacy classes of those subgroups are in one to one correspondence with unpointed trivalent diagrams. We also give in the form of generating series, the number of those trivalent diagrams both pointed or not, as well as generating series for both rooted and unrooted versions of triangular maps up to isomorphism. That later results was a difficult open problem.

Introduction

The theory of combinatorial maps is a venerable subject dating back to Cayley and Hamilton. Since those time it generated an impressive amount of results of all sorts concerning the enumeration problem of counting the *rooted* combinatorial maps. Those results came from various communities of researchers, each with its own methods and tradition. Among them, *enumerative combinatorists* of course played a significant role, starting with pioneering works by Tutte [16] on rooted planar maps. *Theoretical physicists* also played a significant role, starting with the work by t'Hooft [15] on integrals on random matrix spaces. Pure mathematicians like Harer and Zagier [7] also have contributed to the theory in connection with cutting edge algebraic geometry problems concerning modular spaces of Riemann surfaces. Last but not least, one must mention in mathematical physics the Witten-Kontsevich model of quantum gravity [9] using in a central fashion the higher combinatorics of triangular maps and trivalent diagrams.

Although *a lot* is known concerning the theory of *rooted* combinatorial maps, *very little* is known concerning the outstanding problem of enumeration of *unrooted* combinatorial maps up to isomorphism. It appears as a very difficult problem of combinatorics, which stayed barely untouched for almost 150 years. As a matter of fact, the only general result on those important objects were up to now contained in the recent paper by A. Mednykh

Key words and phrases: rooted triangular maps, unrooted triangular maps, modular group, combinatorial group theory, enumerative combinatorics, generating series, cycle index series.

Submitted to STACS (Symposium on Theoretical Aspects of Computer Science)

and R. Nedela [10]. At the end of this paper, we give our first contribution to this difficult problem, namely in the form of a generating series giving the number of *unrooted* triangular maps (*c.f.* table 5 on page 9).

The aim of this paper is to give a preliminary overview of some of the results of the thesis prepared by the author at the Laboratoire d'Informatique Fondamentale de Lille (LIFL) under the direction of M. Petitot and M. Huttner. Among other results we propose in this thesis a classification of the subgroups, and their conjugacy classes, in a free product of cyclic groups like $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$. Examples of such groups are :

- Free groups on 2,3,..., n generators.
- The *Modular Group* $\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.
- Various *Cartographic Groups* like $\mathscr{C}_2^+ \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$.
- The *Hecke Groups* $\mathscr{H}_n \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$.
- Etc..

Concerning *diagrams* and *subgroups*, we obtained :

- Beautiful structures (*trivalent diagrams*, *triangular maps*, etc...).
- A fully explicit and computable classification.
- Counting principles.
- A new way to compute cycle index series.
- A new family of CAT algorithms to generate the examples.
- A fascinating connection with *Combinatorial Maps*.

Concerning *combinatorial maps*, we obtained :

- Generating series of all sorts : $\tilde{M}_3^\bullet(t)$, $\tilde{M}_3(t)$, $\tilde{M}^\bullet(u_1, u_2, \ldots; t)$, $\tilde{M}(u_1, u_2, \ldots; t)$ etc...
- Recurrence relations on coefficients and differential equations.
- A *recursive decomposition* of triangular maps.
- An *unbiased random sampler* as well as an *exhaustive generator*.
- A new fascinating connection with the *Airy function*.

For the sake of simplicity, we shall focus in the present article on the *modular group* and on *triangular maps*.

1. Trivalent Diagrams

Definition 1.1. *Trivalent Diagrams* are bicolored $\{\bullet, \circ\}$ connected graphs with cyclic orientation and degree conditions at the vertices.

- *Black* ($\bullet$) vertices are either *trivalent* or *univalent*.
- *White* ($\circ$) vertices are either *bivalent* or *univalent*.
- Trivalent black vertices are cyclically oriented.

Definition 1.2. A *morphism* φ between two trivalent diagrams is a collection of three maps $\varphi_\bullet$, $\varphi_\circ$ and φ_- sending the *black vertices*, *white vertices*, and the *edges* of the first diagram to corresponding elements of the second diagram, preserving *adjacencies* and *cyclic orientations*.

Example 1.3. Trivalent diagrams are conveniently represented graphically as little drawings like the ones below. The cyclic orientation around the black vertices are rendered implicit by adopting the standard counter clockwise orientation of the figure. For example,

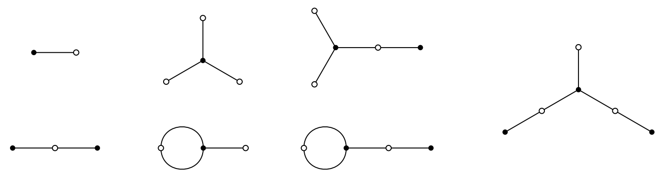


Figure 1. Trivalent diagrams of size up to *five*.

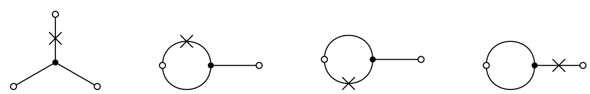


Figure 2. Pointed trivalent diagrams of size *three*

the two following diagrams are considered essentially different even if they are identical as plain colored graphs.

Definition 1.4. *Pointed Trivalent Diagrams* are trivalent diagrams with a distinguished edge, which is called the *base point* of the diagram (*sic)*.

Definition 1.5. *Pointed morphisms* are supposed to send *base points* to *base points*.

Example 1.6. Figure 2 gives the complete list of all pointed trivalent diagrams of size three. One sees on this example that the number of different ways of pointing a trivalent diagram heavily depends on the number of its automorphisms. This exemplify the statement that the unlabeled number of pointed trivalent diagrams bears no useful relation with that of unpointed trivalent diagrams. The same is true for combinatorial maps. Rooted combinatorial maps are well understood combinatorial objects whereas very little is known about the unlabeled number of unrooted ones.

1.1. A Fully Explicit and Computable Classification

Let's recall briefly some very basic facts about the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

Definition 1.7. The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is the group of integer matrices with unit determinant,

$$\mathrm{PSL}_2(\mathbb{Z}) := \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}) / \pm \mathrm{Id} \,\middle|\, ad - bc = 1 \right\} \tag{1.1}$$

The modular group acts on the Poincaré upper half-plan

$$\mathscr{H} = \{ \tau \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0 \} \tag{1.2}$$

by the following homographic transformations,

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau \overset{\text{def.}}{=} \frac{a\,\tau + b}{c\,\tau + d} \tag{1.3}$$

There are many possible finite presentations for this group and we shall stick to the following,

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle\, A, B \mid A^2 = B^3 = 1 \,\rangle \tag{1.4}$$

with A and B being the following two matrices,

$$A = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \pm\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5}$$

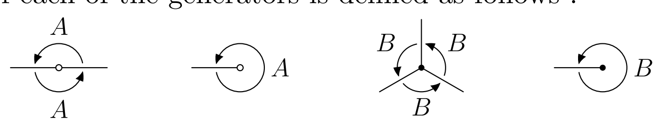
for it renders explicit the following isomorphism,

$$\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \tag{1.6}$$

The elements of $\mathrm{PSL}_2(\mathbb{Z})$ can be seen as words on the two letters A and B with no *two* consecutive A nor *three* consecutive B. The language of elements in normal form is rational. Computations can be maid in the free monoid on the four letters A, $\bar{A}$, B and $\bar{B}$, with the following confluent system of rewriting rules :

$$\bar{A} \to A, \qquad A^2 \to 1, \qquad \bar{B} \to B^2, \qquad B^3 \to 1 \tag{1.7}$$

1.1.1. *Trivalent Diagrams as Homogeneous Spaces of* $\mathrm{PSL}_2(\mathbb{Z})$. The modular group acts transitively on the set of edges of any connected trivalent diagram. This action is *right sided*. The action of each of the generators is defined as follows :



Theorem 1.8 (Classification Principle)**.**

- *The* subgroup *described by a* pointed trivalent diagram *is the isotropy subgroup of the distinguished edge, and this correspondence is* biunivoque*.*
- *As the basepoint is moved, the isotropy subgroup gets conjugated. This way, to any* unpointed trivalent diagram *one associate a* conjuguacy classe of subgroups*. This correspondence is again* biunivoque*.*

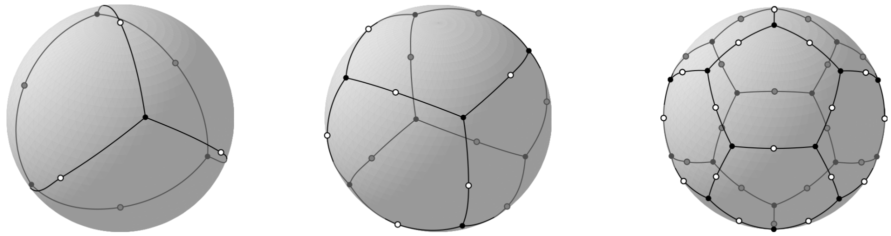
Demonstration. See [22]. □

1.2. A Series of Examples from Classical Mathematics

Using the correspondence of the previous theorem one can ask what are the trivalent diagrams associated to some well known subgroups of the modular group. The modular groups of level n Γ_n, constitute for example an infinite list of such subgroups. They are consisting of matrix (*up to sign*) congruent to the identity modulo n. We have the following short exact sequence.

$$0 \longrightarrow \Gamma_n \longrightarrow \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \longrightarrow 0 \tag{1.8}$$

It was a great surprise to find that the trivalent diagrams associated to the first of those subgroups are the familiar platonic solids depicted in figure 3. For $n = 7$ one recovers the



$n = 3$: the *tetrahedron*. $n = 4$: the *cube*. $n = 5$: the *dodecahedron*.

Figure 3. The trivalent diagrams associated by theorem 1.8 to the modular groups of level three, four and five are familiar platonic solids.

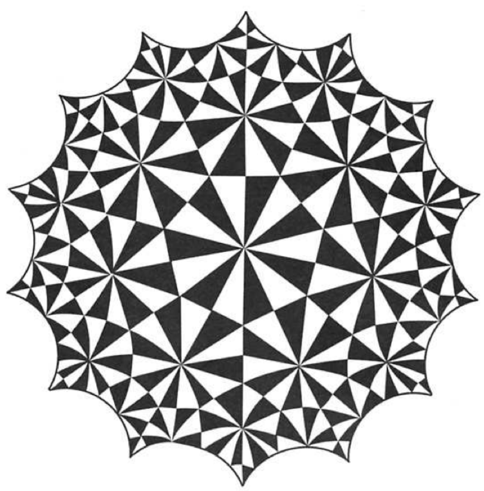


Figure 4. $n = 7$: Klein's quartic.

celebrated Klein's quartic $x^3y + y^3z + z^3x = 0$ with its 168 automorphisms. Wonder never cease...

2. Counting Principles

The species D_3^* of *not necessarily connected* trivalent diagrams is isomorphic to the *direct product* of the species S_2 and S_3 of permutations of compositional order *two* and *three*.

$$D_3^* \underset{nat.}{\simeq} S_2 \times S_3 \tag{2.1}$$

The species D_3 of *connected* trivalent diagrams is related to D_3^* by the following isomorphism.

$$D_3^* \underset{nat.}{\simeq} \mathrm{Set}(D_3) \tag{2.2}$$

	Pointed	Not pointed
Labeled	$D_3^\bullet(t) = \sum_{n\geq 0} \frac{a_n^\bullet}{n!}\, t^n$	$D_3(t) = \sum_{n\geq 0} \frac{a_n}{n!}\, t^n$
Unlabeled	$\tilde{D}_3^\bullet(t) = \sum_{n\geq 0} \tilde{a}_n^\bullet\, t^n$	$\tilde{D}_3(t) = \sum_{n\geq 0} \tilde{a}_n\, t^n$

Table 1. Notations for the generating series associated to trivalent diagrams.

Expressing the *existence* and *uniqueness* of the decomposition of a diagram in its *connected components.*

Following the general paradigm in enumerative combinatorics telling the precise correspondence between the combinatorial operations on structures and the algebraic operations on their associated generating series, we deduce from the two combinatorial isomorphisms 2.1 and 2.2 the following equations,

- On labelled generating series,

$$S_2(t) = \exp\left(t + \frac{t^2}{2}\right) \tag{2.3}$$

$$S_3(t) = \exp\left(t + \frac{t^3}{3}\right) \tag{2.4}$$

$$D_3^*(t) = S_2(t) \odot S_3(t) \tag{2.5}$$

$$D_3(t) = \log(D_3^*(t)) \tag{2.6}$$

- On cycle index series,

$$\mathcal{Z}_{S_2}(x_1, x_2, \dots) = \exp \sum_{k\geq 0} \left(\frac{x_k}{k} + \frac{x_k^2}{2k} + \frac{x_{2k}}{2k}\right) \tag{2.7}$$

$$\mathcal{Z}_{S_3}(x_1, x_2, \dots) = \exp \sum_{k\geq 0} \left(\frac{x_k}{k} + \frac{x_k^3}{3k} + \frac{x_{3k}}{3k}\right) \tag{2.8}$$

$$\mathcal{Z}_{D_3^*}(x_1, x_2, \dots) = \mathcal{Z}_{S_2}(x_1, x_2, \dots) \odot \mathcal{Z}_{S_3}(x_1, x_2, \dots) \tag{2.9}$$

$$\mathcal{Z}_{D_3}(x_1, x_2, \dots) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \mathcal{Z}_{D_3^*}(x_k, x_{2k}, \dots) \tag{2.10}$$

Where the $\odot$ symbols in formulae 2.5 and 2.9 are suitable variation of the Hadammard product. On general exponential series $A(t)$ and $B(t)$ given by,

$$A(t) = \sum_{n\geq 0} \frac{a_n}{n!} t^n \quad \text{and} \quad B(t) = \sum_{n\geq 0} \frac{b_n}{n!} t^n \tag{2.11}$$

$$\begin{aligned}\tilde{D}_3^\bullet(t) \;=\; & t + t^2 + 4\,t^3 + 8\,t^4 + 5\,t^5 + 22\,t^6 + 42\,t^7 + 40\,t^8 \\ & + 120\,t^9 + 265\,t^{10} + 286\,t^{11} + 764\,t^{12} + 1729\,t^{13} \\ & + 2198\,t^{14} + 5168\,t^{15} + 12144\,t^{16} + 17034\,t^{17} \\ & + 37702\,t^{18} + 88958\,t^{19} + 136584\,t^{20} + 288270\,t^{21} \\ & + 682572\,t^{22} + 1118996\,t^{23} + 2306464\,t^{24} \\ & + 5428800\,t^{25} + 9409517\,t^{26} + 19103988\,t^{27} \\ & + 44701696\,t^{28} + 80904113\,t^{29} + 163344502\,t^{30} + \dots\end{aligned}$$

Table 2. The number of *pointed trivalent diagrams* (A005133), which is also, using the theorem 1.8, the number of subgroups of the given index in the modular group.

it is defined by,

$$A(t) \odot B(t) \overset{\text{def.}}{=} \sum_{n\geq 0} \frac{a_n\, b_n}{n!} t^n \tag{2.12}$$

On general cycle index series $\mathcal{Z}_A$ and $\mathcal{Z}_B$ given by,

$$\mathcal{Z}_A = \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{a_{k_1,\dots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n} \text{ and} \tag{2.13}$$

$$\mathcal{Z}_B = \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_{k_1,\dots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n} \tag{2.14}$$

it is defined by,

$$\mathcal{Z}_A \odot \mathcal{Z}_B \overset{\text{def.}}{=} \sum_{n\geq 0} \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{a_{k_1,\dots,k_n}\, b_{k_1,\dots,k_n}}{1^{k_1}\, k_1! \cdots n^{k_n}\, k_n!}\, x_1^{k_1} \cdots x_n^{k_n} \tag{2.15}$$

Rigidity Principle. Pointed trivalent diagrams don't have any automorphism. Thus, *ordinary* and *exponential* generating series coincide. We finally get,

$$\tilde{D}_3^\bullet(t) = D_3^\bullet(t) = t\, \frac{\partial}{\partial t}\, D_3(t) \tag{2.16}$$

Enabling this way, an *unlabeled* counting from a *labeled* one.

The result of the computation is displayed in table 2 and changing slightly the definition of the species S_2 and S_3 to that of S_2^+ and S_3^+ of the permutations having order two and three and no fixed points, then one obtain the corresponding generating series for the species of *pointed triangular maps* shown in table 3.

Recurrence Relation. If we note b_n the coefficient of t^{6n} in $\tilde{M}_3^\bullet(t)$ (*c.f.* table 3) the recurrence is as follows,

$$b_1 = 5 \quad \text{and} \quad b_{n+1} = (6n+6)\, b_n + \sum_{k=1}^{n-1} b_k\, b_{n-k} \quad (n \geq 1) \tag{2.17}$$

$$\begin{aligned}\tilde{M}_3^{\bullet}(t) &= 5\,t^6 + 60\,t^{12} + 1105\,t^{18} + 27120\,t^{24} + 828250\,t^{30} \\ &\quad + 30220800\,t^{36} + 1282031525\,t^{42} + 61999046400\,t^{48} \\ &\quad + 3366961243750\,t^{54} + 202903221120000\,t^{60} + \dots\end{aligned}$$

Table 3. The number of *pointed triangular maps* on an closed orientable surface (A062980). (**New !**)

2.1. A Fascinating New Connection with Airy Asymptotics

We discovered an unexpected an very beautiful connection between some of those generating series and the asymptotic expansion of the Airy function,

$$\mathrm{Ai}(x) = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} e^{i(xt+t^3/3)}dt \tag{2.18}$$

Due to lake of space we can only scratch the surface of those deep connections but let tell for instance that $D_3^*(t)$ is an hypergeometric series of Gevray order one,

$$D_3^*(t) = {}_2\mathrm{F}_0\left(\begin{matrix}\frac{1}{6},\frac{5}{6}\\ -\end{matrix}\middle|\, 6\,t^6\right) = \sum_{k=0}^{n}\frac{(\frac{1}{6})_k(\frac{5}{6})_k}{k!}(6\,t^6)^k \tag{2.19}$$

which shows up in the asymptotic expansion of the Airy function,

$$\mathrm{Ai}(x) \underset{x\to+\infty}{\sim} \frac{e^{-2/3x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_k(\frac{5}{6})_k}{k!}\left(-\frac{3}{4}x^{-3/2}\right)^k \tag{2.20}$$

$$= \frac{e^{-2/3x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}\left(1 - \frac{5}{48}\,x^{-3/2} + \frac{385}{4608}\,x^{-3} - \frac{85085}{663552}\,x^{-9/2} + \dots\right) \tag{2.21}$$

The coefficients b_n appearing in the previous recurrence relation 2.17 also gives the exact coefficients of the following asymptotic development of the logarithmic derivative :

$$\frac{\mathrm{Ai}'(x)}{\mathrm{Ai}(x)} \underset{x\to+\infty}{\sim} -\sqrt{x} - \frac{1}{4x} - \sum_{k\geq 1}(-1)^k\,a_k\,(4\,x)^{\frac{2-3k}{2}} \tag{2.22}$$

$$\begin{aligned} &= -\sqrt{x} - \frac{1}{4}\,x^{-1} + \frac{5}{32}\,x^{-5/2} - \frac{15}{64}\,x^{-4} + \frac{1105}{2048}\,x^{-11/2} \\ &\quad - \frac{1695}{1024}\,x^{-7} + \frac{414125}{65536}\,x^{-17/2} + \dots \end{aligned} \tag{2.23}$$

2.2. A New Way to Compute Cycle Index Series

A *general* cycle index series, or Joyal-Pólya series, has the following form.

$$\mathcal{Z}(x_1, x_2, \dots) = \sum_{n\geq 0}\ \sum_{k_1+2k_2+\dots+nk_n=n}\frac{a_{k_1,\dots,k_n}}{1^{k_1}\,k_1!\cdots n^{k_n}\,k_n!}\,x_1^{k_1}\cdots x_n^{k_n} \tag{2.24}$$

where the *degree* of the variables x_k is taken to be k. Those are rather complex objects not very practical for implementation. The general series contains exactly p_n terms in its n-th *graduation* and $p_1 + p_2 + \dots + p_n$ terms in its n-th *filtration*. This is BIG ! *e.g.* more than

$$\begin{aligned}\tilde{D}_3(t) \;=\;& t + t^2 + 2\,t^3 + 2\,t^4 + t^5 + 8\,t^6 + 6\,t^7 + 7\,t^8 + 14\,t^9 \\ & + 27\,t^{10} + 26\,t^{11} + 80\,t^{12} + 133\,t^{13} + 170\,t^{14} + 348\,t^{15} \\ & + 765\,t^{16} + 1002\,t^{17} + 2176\,t^{18} + 4682\,t^{19} + 6931\,t^{20} \\ & + 13740\,t^{21} + 31085\,t^{22} + 48652\,t^{23} + 96682\,t^{24} \\ & + 217152\,t^{25} + 362779\,t^{26} + 707590\,t^{27} + 1597130\,t^{28} \\ & + 2789797\,t^{29} + 5449439\,t^{30} + \dots\end{aligned}$$

Table 4. The number of *trivalent diagrams* (A121350), also by theorem 1.8 the number of *conjugacy classes of subgroups* in the modular group. (**New !**)

$$\begin{aligned}\tilde{M}_3(t) \;=&\; 3\,t^6 + 11\,t^{12} + 81\,t^{18} + 1228\,t^{24} + 28174\,t^{30} + 843186\,t^{36} \\ & + 30551755\,t^{42} + 1291861997\,t^{48} + 62352938720\,t^{54} \\ & + 3381736322813\,t^{60} + 203604398647922\,t^{66} \\ & + 13475238697911184\,t^{72} + 972429507963453210\,t^{78} \\ & + 75993857157285258473\,t^{84} \\ & + 6393779463050776636807\,t^{90} + \dots\end{aligned}$$

Table 5. The number of *unlabeled triangular maps* on an connected closed orientable surface (A129114). (**New !**)

a million terms in the 50-th filtration. We nevertheless achieved to compute in high weight (up to thousandth order) in less than an hour on a personal computer. The trick come from the following notion.

Definition 2.1. A Joyal-Pólya series is said to be *separated* if it admits an expression of the following form,

$$\mathcal{Z}(x_1, x_2, \dots) = \prod_{k\geq 1} \left(\sum_{n\geq 0} \frac{a_{k,n}}{k^n\, n!}\, x_k^n \right) \quad \text{with } a_{k,0} = 1 \text{ for all } k \geq 1. \tag{2.25}$$

Let's look at the *complexity*. The general series in the *factored form* is very *sparse*. It contains $n + n/2 + n/3 + \cdots = O(n \log n)$ terms in its n-th *filtration*.

If we can do the computations in this representation avoiding the expanded form 2.24 then this produce a enormous collapse in complexity. Roughly specking, this trick turns an almost exponential $\sim e^{\sqrt{n}}$ complexity computation into an almost linear one $O(n \log(n))$ The miracle comes from the following lemma, which is immediate to prove.

$$\begin{aligned}\tilde{M} &= (u_2 + {u_1}^2)\,t + ({u_1}^2 u_2 + u_1 u_3 + {u_2}^2 + 2\,u_4)\,t^2 \\ &+ \begin{pmatrix} u_3 {u_1}^3 + {u_1}^2 {u_2}^2 + {u_2}^3 + 3\,u_1 u_5 + 2\,{u_1}^2 u_4 \\ + 5\,u_6 + 2\,u_2 u_4 + 2\,u_3 u_1 u_2 + \underline{3\,{u_3}^2} \end{pmatrix} t^3 \\ &+ \begin{pmatrix} 9\,{u_1}^2 u_6 + {u_1}^2 {u_2}^3 + 9\,u_2 u_6 + 7\,u_5 u_3 + 2\,u_5 {u_1}^3 \\ + 9\,u_5 u_1 u_2 + 4\,{u_2}^2 u_4 + {u_1}^4 u_4 + 3\,{u_1}^2 {u_3}^2 + {u_2}^4 \\ + 7\,{u_4}^2 + 18\,u_8 + 5\,{u_1}^2 u_2 u_4 + 4\,{u_3}^2 u_2 + 15\,u_1 u_7 \\ + 8\,u_4 u_1 u_3 + u_3 {u_1}^3 u_2 + 3\,u_3 u_1 {u_2}^2 \end{pmatrix} t^4 + \dots\end{aligned}$$

Table 6. The number of *unlabeled combinatorial maps* on an closed orientable surface with a given number n_e of edges (coeff. of t^{n_e}) and a given number n of k-gones (coeff. of u_k^n) . (**New !**)

Lemma 2.2. *Let,*

$$\mathcal{Z}_1 = \prod_{k\geq 1}\left(\sum_{n\geq 0}\frac{a_{k,n}}{k^n\,n!}\,x_k^n\right) \qquad \textit{and} \qquad \mathcal{Z}_2 = \prod_{k\geq 1}\left(\sum_{n\geq 0}\frac{b_{k,n}}{k^n\,n!}\,x_k^n\right) \tag{2.26}$$

are two cycle index series in factored *form, then,*

$$\mathcal{Z}_1 \odot \mathcal{Z}_2 = \prod_{k\geq 1}\left(\sum_{n\geq 0}\frac{a_{k,n}\,b_{k,n}}{k^n\,n!}\,x_k^n\right) \tag{2.27}$$

The results of those computations are given in table 4 and 5. Table 4 gives the exact number of trivalent diagrams up to isomorphism having a prescribed number of egdes. This is also the number of subgroups of the modular group up to conjugacy having as index that prescribed number of edges. Table 5 gives the exact number of triangulations of connected compact orientable surfaces up to an orientation preserving diffeomorphism of that underlying surface.

Using the same pinciples we have computed a generating series (shown in table 6) giving the number of combinatorial maps on a closed surface with a given number of edges and the list of degree for its faces (or by Poincaré duality, the list of degree for its vertices). It comes thus with an infinite set of parameters t, u_1, u_2, u_3 and the coefficient of $t^{n_e} u_1^{n_1} u_2^{n_2} u_3^{n_3}$... is the number of combinatorial maps with n_e distinct edges and with n_1, n_2, n_3, etc, for the number of its *loops*, *spindles*, *triangles*, etc, or by Poincaré duality, n_1, n_2, n_3, etc for its number of vertices having degree 1, 2, 3, etc..

References

[1] G. V. Belyi. On galois extensions of a maximal cyclotomic field. *Math. USSR Izvestija*, 14(2):247–256, 1980.

[2] Klein F. *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree*. Dover New York, 1956. 2nd and rev. ed.

[3] P. Leroux F. Bergeron, G. Labelle. *Théorie des espèces et combinatoire des structures arborescentes.* LACIM Montréal, 1994.

[4] P. Leroux F. Bergeron, G. Labelle. *Combinatorial Species and Tree-like Structures.* Cambridge University Press, 1998. English edition of [3].

[5] E.M. Palmer F. Harary. *Graphical Enumeration.* Academic Press New York/London, 1973.

[6] A. Grothendieck. Esquisse d'un programme. In P. Lochak L. Schneps, editor, *Geometric Galois Actions Vol. I*, number 242 in London Math. Soc. Lecture Notes, pages 5–48. Cambridge Univ. Press, 1997.

[7] J. Harer and D. Zagier. The euler characteristic of the moduli space of curves. *Invent. Math.*, 85:457–486, 1986.

[8] A. Joyal. Une théorie combinatoire des séries formelles. *Adv. Math.*, 42:1–82, 1981.

[9] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. *Commun. Math. Phys.*, 147:1–23, 1992.

[10] A. Mednykh and R. Nedela. Enumeration of unrooted maps of a given genus. *J. Comb. Theory, Ser. B*, 96(5):706–729, 2006.

[11] P. E. Schupp R. C. Lyndon. *Combinatorial Group Theory.* Classics in Mathematics. Springer-Verlag, 2001. Reprint of the 1977 edition.

[12] J.P. Serre. *Trees.* Springer Monographs in Math. Springer-Verlag, 2003.

[13] A.K. Zvonkin S.K. Lando. *Graphs on Surfaces and Their Applications.* Springer-Verlag, 2004.

[14] N. J. A. Sloane. The on-line encyclopedia of integer sequences. Available on the net at : http://www.research.att.com/∼njas/sequences/, 2005.

[15] G. t'Hooft. A planer diagram theory for strong interactions. *Nuclear Physics B*, 72:461–473, 1974.

[16] W. T. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.

[17] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[18] W. T. Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968.

[19] W. T. Tutte. The enumerative theory of planar maps. In J. N. Srivastava, editor, *A Survey of Combinatorial Theory*, pages 437–448. North-Holland, 1973.

[20] S. A. Vidal. Asymptotics of the Airy Function and Triangular Maps Decomposition. In preparation.

[21] S. A. Vidal. Multiparametric Enumeration of Unrooted Combinatorial Maps. In preparation.

[22] S. A. Vidal. Sur la classification et le dénombrement des sous-groupes du groupe modulaire et de leurs classes de conjugaison. *Pub. IRMA, Lille*, 66(II):1–35, 2006. Preprint : http://arxiv.org/abs/math.CO/0702223.

[23] S. A. Vidal. An Optimal Algorithm to Generate Pointed Trivalent Diagrams an Pointed Triangular Maps. Preprint : http://arxiv.org/abs/0706.0969, 2007.